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LOCAL AND OVERALL EXTREMAL PROPERTIES OF TIME-INDEPENDENT MATERIALS AND NON-LINEAR ELASTIC COMPARISON MATERIALS

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Abstract—For time-independent materials which undergo *non-linear* deformations from some given reference configuration two (dual) hypotheses are considered. Firstly it is supposed that the work done to a given state of deformation is bounded below and that the bound is attainable on a physically possible path; secondly that the complementary work to a given state of stress is bounded above and that this bound too is attainable on a physically possible path. The consequences of these assumptions are analysed, and the results of Ponter and Martin[1] in the linear theory are generalized to account for non-linear deformations, due attention being paid to questions of stability.

A non-linear elastic *comparison* material is defined whose strain energy is equal to the work done on a minimum path for the time-independent material. Extremum principles for non-linear elastic materials given in [2] are then applied to the comparison elastic material, and bounds are thereby placed on the work and complementary-work functionals of the time-independent material. Corresponding overall properties of the time-independent and elastic materials are then compared by defining respective *overall* constitutive laws and overall stress and deformation variables.

Following the definition of strengthening (weakening) of a non-linear elastic solid given by Ogden[2] a time-independent material is said to be strengthened (weakened) when its comparison elastic material is strengthened (weakened). Local and overall aspects of this definition are examined.

1. INTRODUCTION

The question of what mathematical restrictions should be placed on the constitutive laws of solid materials so as to ensure physically reasonable response, and thereby delimit the choice of possible laws, has received considerable attention. It has arisen in several guises, but notably in elasticity theory where the question was raised by Truesdell[3].

For elastic materials Coleman and Noll[4] proposed an inequality which asserts a lower bound to the work of deformation on any path between two prescribed strains. Several authors have investigated the consequences of the Coleman-Noll hypothesis which, as demonstrated by Hill[5] and Ogden[6, 7], is not universally viable. A satisfactory answer to the problem for elastic materials has been found by Hill[5, 8]. It is not our primary intention in this paper to discuss elastic materials; we are interested more in the properties of time-independent materials generally, and on what restrictions it is sensible to place on their large-strain response. However, it transpires that certain properties of time-independent materials can be compared with those of a closely related elastic material and, although this point is not alluded to further in this paper, the properties of this "comparison" elastic material must conform with inequalities which ensure physically sensible mechanical behaviour. The restrictions on the comparison elastic material automatically place restrictions on the constitutive law of the time-independent material.

For time-independent plastic materials Drucker[9, 10] proposed inequalities (similar in form to the Coleman-Noll hypothesis but in the linear context) which assert a lower bound to the work of *plastic* deformation on any path between two prescribed strains. However, like the bound in the Coleman-Noll hypothesis that proposed by Drucker is not attainable on a physically possible path. Hence, Drucker's hypothesis is interpreted as a criterion of stability rather than as a material characteristic. A related inequality was proposed by Ilyushin[11] for time-independent materials. This asserts that the work done in any strain *cycle* is positive when plastic deformation occurs.

Hill[5, 12] has investigated the consequences and interpretations of these and other inequalities in the context of a generalized class of stress and strain variables, namely *conjugate* variables (see also the recent paper by Hill and Rice[13]).

A variant of the above inequalities for infinitesimal strains has been discussed by Ponter[14] following Martin[15] who postulated that the *complementary* work done between any two prescribed stresses is bounded above. Their approach differs from that of other writers in that the bound is not at first stated explicitly, but is assumed to be attainable on a physically possible path. A similar statement can be made about the minimum work between prescribed strains. Ponter and Martin[1] have extended the results given in[14]. For a given constitutive law the work and complementary-work bounds can in principle be determined by calculating the extremal paths in strain and stress space and, although this is difficult in practice, Ponter and Martin[1] have given specific examples of such paths for some particular plastic constitutive laws.

Ponter[14] has shown that the upper bound to the complementary work done between prescribed stresses is a convex function of the final state of stress provided the scalar product of stress-rate and strain-rate is always positive (semi-) definite. This result was proved strictly in the linear context so there was no need to distinguish between different (conjugate) measures of strain or stress. Ponter and Martin[1] have demonstrated that a path in strain space which corresponds to a maximum complementary work path in stress space is a minimum work path. A converse of this result is proved in the Appendix, although there is not a complete duality between the two results. The above results are easily generalized to the non-linear theory within certain limitations. In fact, Ponter and Martin's results had in part been proved earlier by Ogden[16] for non-linear strains and in terms of conjugate variables.

In the present paper Ponter and Martin's results are first of all extended to the non-linear theory following a brief resumé of elasticity theory in Section 2, it being appropriate to work in terms of the conjugate nominal stress and displacement gradient. Attention must be restricted to deformations from some reference configuration and within a certain domain of deformation space—essentially the domain for which a "comparison" elastic material (defined in Section 3) is stable under dead loading.

For inhomogeneous materials detailed knowledge of the interior stress and deformation fields is not in general available but for particular boundary-value problems the average stress or deformation throughout the material can be expressed in terms of prescribed surface data. Information as to the overall characteristics of the material can thereby be obtained by working in terms of such averaged quantities and a corresponding overall constitutive law.

An overall time-independent constitutive law and a comparison overall elastic constitutive law may each be defined, and, by means of extremum principles, their properties related in a way analogous to that in which local properties are related. These results are discussed in Section 4.

Finally, in Section 5, the definition of strengthening of an elastic material given in [2] is adopted to provide a definition of strengthening for time-independent materials. The paper is theoretical in nature, detailed considerations of the theory for particular materials being reserved for a separate report.

2. PRELIMINARY ELASTICITY THEORY

Suppose that $\mathbf{X}$ and $\mathbf{x}$ respectively denote the position vectors of a typical material point in some reference configuration and the current configuration, and let $\boldsymbol{\alpha}$ denote the displacement gradient relative to the reference configuration.

Consider an elastic material which has strain energy $W \equiv W(\boldsymbol{\alpha})$ per unit reference volume. Let the material occupy volume V, bounded by surface Σ, in the reference configuration. For this material the nominal stress tensor $\mathbf{s}$ is given by

$$\mathbf{s} = \frac{\partial W}{\partial \boldsymbol{\alpha}}. \tag{1}$$

The traction $\mathbf{t}$ per unit area on a surface element normal to the unit vector $\mathbf{n}$ in the reference configuration may be written as

$$\mathbf{t} = \mathbf{s}^T \mathbf{n}, \tag{2}$$

where $\mathbf{s}^T$ is the transpose of $\mathbf{s}$.

When there are no body forces (they may easily be incorporated if required) the equilibrium equations are written

$$\operatorname{div} \mathbf{s} = 0, \tag{3}$$

where the divergence is with respect to $\mathbf{X}$.

We suppose that the displacement $\mathbf{u}$ and traction $\mathbf{t}$ respectively are prescribed on the complementary parts Σ_u and Σ_s of the surface Σ, so that

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Sigma_u, \tag{4}$$

$$\mathbf{t} = \mathbf{t}_0 \quad \text{on} \quad \Sigma_s, \tag{5}$$

$\mathbf{u}_0$ and $\mathbf{t}_0$ being given functions.

The complementary energy $W_c \equiv W_c(\mathbf{s})$ relative to the conjugate variables $\mathbf{s}$ and $\boldsymbol{\alpha}$ is defined uniquely by

$$W + W_c = \mathbf{s}\boldsymbol{\alpha} \tag{6}$$

when the relationship $\mathbf{s} \equiv \mathbf{s}(\boldsymbol{\alpha})$ is uniquely invertible (locally), as it is in some domain enclosing the undeformed configuration of the body provided pure rotations are disregarded. The quantity $\mathbf{s}\boldsymbol{\alpha}$ denotes the scalar product (i.e. the trace of the matrix product of $\mathbf{s}$ and $\boldsymbol{\alpha}$) and the scalar product of two vectors is denoted similarly.

It follows that

$$\boldsymbol{\alpha} = \frac{\partial W_c}{\partial \mathbf{s}}, \tag{7}$$

uniquely (locally) in the *primary domain* $\mathscr{D}$, that is the domain in $\boldsymbol{\alpha}$-space where $\partial \mathbf{s}/\partial \boldsymbol{\alpha}$ is positive definite[2]. Within $\mathscr{D}$ the inequality

$$\dot{\mathbf{s}}\dot{\boldsymbol{\alpha}} > 0 \tag{8}$$

holds for arbitrary $\dot{\boldsymbol{\alpha}}$, the dot denoting the time rate at constant $\mathbf{X}$. In $\mathscr{D}$, $\boldsymbol{\alpha}$ given by (7) is not necessarily unique in the global sense since non-uniqueness is possible if $\mathscr{D}$ is not a convex domain.

Explicit dependence of W on $\mathbf{X}$ has not been indicated above, but it may be understood for inhomogeneous materials.

Energy and complementary-energy functionals $E(\boldsymbol{\alpha})$ and $E_c(\mathbf{s})$ are defined by

$$E(\boldsymbol{\alpha}) = \int_V W(\boldsymbol{\alpha})\,\mathrm{d}V - \int_{\Sigma_s} \mathbf{u}\mathbf{t}_0\,\mathrm{d}\Sigma, \tag{9}$$

and

$$E_c(\mathbf{s}) = \int_{\Sigma_u} \mathbf{t}\mathbf{u}_0\,\mathrm{d}\Sigma - \int_V W_c(\mathbf{s})\,\mathrm{d}V. \tag{10}$$

When $\boldsymbol{\alpha}$ and $\mathbf{s}$ correspond to the actual solution of the boundary-value problem it follows from the above equations that

$$E(\boldsymbol{\alpha}) = E_c(\mathbf{s}). \tag{11}$$

3. MINIMUM WORK AND MAXIMUM COMPLEMENTARY WORK

We now consider a time-independent material in equilibrium in some prescribed initial state (which may be taken as the virgin state), the material occupying volume V and bounded by surface Σ. This initial state is taken as the reference configuration from which the deformation is measured.

The material is deformed to a new equilibrium configuration (without body forces), and is subject to the boundary conditions on Σ specified by eqns (4) and (5). The distribution of stress and deformation depends on the constitutive law of the material and, in general, on the particular deformation path taken to achieve this distribution.

The work done in achieving the deformation $\boldsymbol{\alpha}^*$ (locally) is given by

$$\int_0^{\boldsymbol{\alpha}^*} \mathbf{s}\,\mathrm{d}\boldsymbol{\alpha},$$

a quantity which is invariant with respect to change of conjugate variables [8, 12, 19]. For convenience this is denoted by

$$\phi(\boldsymbol{\alpha}^*) = \int_0^{\boldsymbol{\alpha}^*} \mathrm{s}\, d\boldsymbol{\alpha}, \tag{12}$$

although, of course, the integral will in general depend on the path of integration in $\boldsymbol{\alpha}$-space, not just on the terminal point $\boldsymbol{\alpha}^*$.

Dually, if s_0 (which may be zero) denotes the existing state of stress in the reference configuration, the complementary work done (relative to variables s and $\boldsymbol{\alpha}$) in achieving the nominal stress s^* locally is

$$\int_{\mathrm{s}_0}^{\mathrm{s}^*} \boldsymbol{\alpha}\, d\mathrm{s},$$

which is not invariant with respect to change of conjugate variable.

We use the notation

$$\phi_c(\mathrm{s}^*) = \int_{\mathrm{s}_0}^{\mathrm{s}^*} \boldsymbol{\alpha}\, d\mathrm{s} \tag{13}$$

for any given s_0, and this integral in general depends on the path taken in s-space.

Clearly,

$$\phi(\boldsymbol{\alpha}^*) + \phi_c(\mathrm{s}^*) = \mathrm{s}^*\boldsymbol{\alpha}^* \tag{14}$$

when $\begin{Bmatrix} \mathrm{s}^* \\ \boldsymbol{\alpha}^* \end{Bmatrix}$ is the $\begin{Bmatrix} \text{nominal stress} \\ \text{displacement gradient} \end{Bmatrix}$ achieved in following some specific path in $\begin{Bmatrix} \boldsymbol{\alpha} \\ \mathrm{s} \end{Bmatrix}$-space to $\begin{Bmatrix} \boldsymbol{\alpha}^* \\ \mathrm{s}^* \end{Bmatrix}$. Then $\begin{Bmatrix} \phi_c(\mathrm{s}^*) \\ \phi(\boldsymbol{\alpha}^*) \end{Bmatrix}$ is the $\begin{Bmatrix} \text{complementary work} \\ \text{work} \end{Bmatrix}$ along the path in $\begin{Bmatrix} \mathrm{s} \\ \boldsymbol{\alpha} \end{Bmatrix}$-space which is the trace of the given path in $\begin{Bmatrix} \boldsymbol{\alpha} \\ \mathrm{s} \end{Bmatrix}$-space.

Two alternative hypotheses are now available. Firstly, it may be assumed that $\phi(\boldsymbol{\alpha}^*)$ is bounded below and that this bound is attainable on a (not necessarily unique) physically possible path in $\boldsymbol{\alpha}$-space. Let $W(\boldsymbol{\alpha}^*)$ denote the bound. Then

$$\phi(\boldsymbol{\alpha}^*) \geq W(\boldsymbol{\alpha}^*) \tag{15}$$

for all paths to $\boldsymbol{\alpha}^*$ from the origin in $\boldsymbol{\alpha}$-space.

Secondly, following Ponter and Martin [1, 14, 15], it may be assumed that $\phi_c(\mathrm{s}^*)$ is bounded above within the class of nominal stress paths from s_0 to s^*. Let $W_c(\mathrm{s}^*)$ denote the bound so that

$$\phi_c(\mathrm{s}^*) \leq W_c(\mathrm{s}^*), \tag{16}$$

with equality holding on at least one physically possible path in s-space. Since we are dealing with non-linear deformations ϕ_c, and hence W_c, depends on the choice of conjugate variables. It is important in the application of these hypotheses that (s, $\boldsymbol{\alpha}$)-variables are employed. The assumption of smooth behaviour, or equivalently differentiability of $W(\boldsymbol{\alpha})$ and $W_c(\mathrm{s})$ leads to the equations

$$\mathrm{s} = \frac{\partial W}{\partial \boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} = \frac{\partial W_c}{\partial \mathrm{s}}, \tag{17}$$

as proved by Ponter and Martin [1] in the linear context and Ogden [16] for non-linear deformations using general conjugate variables.

The first equation in (17) states that the nominal stress achieved on a minimum work path to $\boldsymbol{\alpha}$

is given uniquely as $s = \partial W/\partial \boldsymbol{\alpha}$ and, correspondingly, the second states that the deformation achieved on a maximum complementary-work path to s is given by $\boldsymbol{\alpha} = \partial W_c/\partial s$. This $\boldsymbol{\alpha}$ is unique provided attention is restricted to a certain domain in $\boldsymbol{\alpha}$-space, as described below. Of course, the paths themselves may not be unique.

These results are dual in the sense that the path traced out in s-space ($\boldsymbol{\alpha}$-space) by a minimum-work (maximum complementary-work) path in $\boldsymbol{\alpha}$-space (s-space) is a maximum complementary-work (minimum work) path. The second of these was proved in [1] for the linear theory using the assumption that $W(\boldsymbol{\alpha})$ is a strictly locally convex function for all $\boldsymbol{\alpha}$ (that is the inequality (8) holds for all $\boldsymbol{\alpha}$, and W and W_c are Legendre duals relative to s and $\boldsymbol{\alpha}$), together with the second hypothesis mentioned above. In this paper we show how the result can be extended to the non-linear theory. In the Appendix we also prove the first result on the basis of the first hypothesis.

In the non-linear theory of elastic deformations the inequality (8) cannot be expected to hold at all points in $\boldsymbol{\alpha}$-space since it is violated in unstable situations. We must therefore confine attention to some domain of $\boldsymbol{\alpha}$-space for which (8) is valid in respect of W. This is the domain of local stability under dead loading. In fact we restrict attention to that part of the domain, $\mathscr{D}$ say, which includes the origin, since the domain may be disjoint. Let $\mathscr{D}_c$ be the domain in s-space to which $\mathscr{D}$ is mapped by $(17)_1$. If $\mathscr{D}$ is a convex domain then the mapping is one to one on $\mathscr{D}$, W_c exists on $\mathscr{D}_c$, and for a given s in $\mathscr{D}_c$ a unique $\boldsymbol{\alpha}$ in $\mathscr{D}$ is specified by $(17)_2$. If $\mathscr{D}$ is not convex the requirement of strict *local* convexity (8) is replaced by one of strict global convexity of W on $\mathscr{D}$.

Since the minimum-maximum duality is valid on $\mathscr{D}$, W and W_c are Legendre duals there and we can write

$$W(\boldsymbol{\alpha}) + W_c(s) = s\boldsymbol{\alpha} \tag{18}$$

for any given reference configuration. This, of course, is just the specialization of (14) to the extremal path situation.

It is now apparent that the extremal paths of the time-independent material characterize the response of a purely elastic material as set out in Section 2. When the extremal paths are known for any given time-independent material we can define an elastic material which has strain energy W per unit reference volume. Then the properties of the time-independent material may be compared with those of the elastic material. Accordingly, we refer to the elastic material whose strain energy is equal to the work done on a minimum work path in the time-independent material as the *comparison elastic material.*

It follows that within $\mathscr{D}$ the comparison elastic material is stable under dead loading. Since W is globally convex in $\mathscr{D}$ we have

$$W(\boldsymbol{\alpha}^*) - W(\boldsymbol{\alpha}) - s(\boldsymbol{\alpha}^* - \boldsymbol{\alpha}) \geq 0, \tag{19}$$

where $s = \partial W(\boldsymbol{\alpha})/\partial \boldsymbol{\alpha}$, for all $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ in $\mathscr{D}$. Correspondingly,

$$W_c(s^*) - W_c(s) - \boldsymbol{\alpha}(s^* - s) \geq 0, \tag{20}$$

where $\boldsymbol{\alpha} = \partial W_c(s)/\partial s$, for all s and s^* in $\mathscr{D}_c$. The equalities in (19) and (20) respectively hold if and only if $\boldsymbol{\alpha}^* \equiv \boldsymbol{\alpha}$ and $s^* \equiv s$.

It should be pointed out here that in [1] the constitutive law of the time-independent material was required to satisfy the inequality $\dot{s}\dot{\boldsymbol{\alpha}} \geq 0$, thus allowing rigid and perfectly plastic behaviour. Here, however, the equality is not conceded.

If required the domains $\mathscr{D}$ and $\mathscr{D}_c$ can be extended by imposing rotational restrictions on $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$, as in [2]. *Indeed, when the form of W is known a restricted form of the inequality* (19) *can be shown to hold for a far larger range of deformations than first envisaged* [2]. From (15) and (19) we have

$$\phi(\boldsymbol{\alpha}^*) \geq W(\boldsymbol{\alpha}^*) \geq W(\boldsymbol{\alpha}) + s(\boldsymbol{\alpha}^* - \boldsymbol{\alpha}), \tag{21}$$

for all $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ in $\mathscr{D}$, where s is given by (17_1), while from (16) and (20)

$$\phi_c(s^*) \leq W_c(s^*) \leq W_c(s) - \boldsymbol{\alpha}^*(s - s^*) \tag{22}$$

for all s and s* in $\mathscr{D}_c$, where $\boldsymbol{\alpha}^*$ is given by $(17)_2$ evaluated at s*. Equation (22) corresponds to Ponter and Martin's eqn (28)[1].

We now return to the boundary-value problem specified at the beginning of this section and define the work functional $\Phi(\boldsymbol{\alpha}^*)$ by

$$\Phi(\boldsymbol{\alpha}^*) = \int_V \phi(\boldsymbol{\alpha}^*)\,\mathrm{d}V - \int_{\Sigma_s} \mathbf{u}^*\mathbf{t}_0\,\mathrm{d}\Sigma \tag{23}$$

in respect of the displacement gradient $\boldsymbol{\alpha}^*$ derivable from the displacement $\mathbf{u}^*$ which satisfies the boundary condition (4).

Correspondingly, the complementary work functional $\Phi_c(\mathbf{s}^*)$ is defined by

$$\Phi_c(\mathbf{s}^*) = \int_{\Sigma_u} \mathbf{t}^*\mathbf{u}_0\,\mathrm{d}\Sigma - \int_V \phi_c(\mathbf{s}^*)\,\mathrm{d}V \tag{24}$$

for any self-equilibrated nominal stress field $\mathbf{s}^*$ satisfying the boundary condition (5).

If, in particular, $\mathbf{s}^*$ and $\boldsymbol{\alpha}^*$ are related variables in the sense implied by eqn (14) then it follows from (3)–(5) and the divergence theorem that

$$\Phi(\boldsymbol{\alpha}^*) = \Phi_c(\mathbf{s}^*). \tag{25}$$

We remark that Ponter and Martin define $-\Phi_c$ as their complementary work functional, and that they did not mention the connexion (25).

Corresponding functionals can be written down in respect of extremal paths or, equivalently, for the comparison elastic material. These are denoted respectively by $E(\boldsymbol{\alpha})$ and $E_c(\mathbf{s})$, and defined by eqns (9) and (10), the identity (11) being satisfied when $\boldsymbol{\alpha}$ and s represent an actual solution of the boundary-value problem for the elastic material.

For elastic materials under non-linear deformation certain extremum principles are available (see [2] and the earlier paper by Hill [17]). These have been discussed in detail in [2]. See also the discussion of related variational theorems [18]. They enable bounds to be put on the energy and complementary energy functionals (9) and (10). In view of the relation between the time-independent material and its comparison elastic material corresponding bounds can be found (as in [1]) on the work and complementary-work functionals of the time-independent material.

Let $\boldsymbol{\alpha}'$ be the displacement gradient corresponding to the displacement vector $\mathbf{u}'$ which satisfies the boundary condition on Σ_u and such that $\boldsymbol{\alpha}'$ is in the primary domain $\mathscr{D}$. It follows by (19), the divergence theorem and the boundary conditions that

$$E(\boldsymbol{\alpha}') \geqslant E(\boldsymbol{\alpha}), \tag{26}$$

that is within the domain $\mathscr{D}$ the functional $E(\boldsymbol{\alpha}')$ is minimised by the actual displacement gradient field $\boldsymbol{\alpha}$.

Dually, for any self-equilibrated nominal stress field s′ satisfying the boundary condition on Σ_s and such that s′ is in $\mathscr{D}_c$, it follows that

$$E_c(\mathbf{s}) \geqslant E_c(\mathbf{s}'). \tag{27}$$

Thus, within $\mathscr{D}_c$, the functional $E_c(\mathbf{s}')$ is maximized by the actual nominal stress field s. Note that s′ and $\boldsymbol{\alpha}'$ are related variables if and only if they correspond to the actual solution of the boundary-value problem (unique in $\mathscr{D}$).

Let $\boldsymbol{\alpha}^*$ be a specific displacement gradient within $\mathscr{D}$ consistent with the boundary condition on Σ_u. Then, by means of (9), (21) and (23), it is easily shown that

$$\Phi(\boldsymbol{\alpha}^*) \geqslant E(\boldsymbol{\alpha}^*).$$

Hence, by (26) with $\boldsymbol{\alpha}' = \boldsymbol{\alpha}^*$,

$$\Phi(\boldsymbol{\alpha}^*) \geqslant E(\boldsymbol{\alpha}), \tag{28}$$

$\boldsymbol{\alpha}$ corresponding to the actual solution of the elastic boundary-value problem. Equation (28) may alternatively be obtained directly from (21) by integration and use of the divergence theorem, boundary conditions and equilibrium equations.

In particular $\boldsymbol{\alpha}^*$ may correspond to an actual solution of the boundary-value problem for the time-independent material. Then let $\boldsymbol{\alpha}^*$ and s^* be related variables for the time-independent material.

The dual result

$$\Phi_c(\mathrm{s}^*) \geq E_c(\mathrm{s}) \tag{29}$$

now follows from (22).

Because of (11) and (25) it is evident that (28) and (29) are equivalent. With the help of (27), therefore, (28) and (29) provide the continued inequalities

$$\Phi(\boldsymbol{\alpha}^*) \equiv \Phi_c(\mathrm{s}^*) \geq E(\boldsymbol{\alpha}^*) \geq E(\boldsymbol{\alpha}) \equiv E_c(\mathrm{s}) \geq E_c(\mathrm{s}') \tag{30}$$

for any self-equilibrated s' in $\mathscr{D}_c$ which satisfies the boundary condition (5).

Ponter and Martin[1] did not point out the connexion (25) which has been employed here to arrive at (30). They stated that their equation (37), equivalent to our inequality

$$\Phi_c(\mathrm{s}^*) \geq E_c(\mathrm{s}) \geq E_c(\mathrm{s}'),$$

provides a bounding theorem for the complementary work functional in terms of any admissible s'. Also that their eqn (41), equivalent to eqn (28) here, provides a bounding theorem for $\Phi(\boldsymbol{\alpha}^*)$ only in terms of the solution of the comparison elastic boundary-value problem.

The extension of Ponter and Martin's results in this direction is made possible by our use of both hypotheses. Ponter and Martin employ just the second hypothesis, which admits the possibility of minimum work paths whose traces are not maximum complementary work paths.

Equation (30) shows clearly that both $\Phi(\boldsymbol{\alpha}^*)$ and $\Phi_c(\mathrm{s}^*)$ are bounded below by $E_c(\mathrm{s}')$ for any admissible s'.

It is not surprising that one cannot in general find a bound on $\Phi(\boldsymbol{\alpha}^*)$ of the form $E(\boldsymbol{\alpha}')$ for any displacement gradient $\boldsymbol{\alpha}'$ in $\mathscr{D}$ satisfying the boundary condition on Σ_u, since this would imply

$$\Phi(\boldsymbol{\alpha}^*) \geq E(\boldsymbol{\alpha}') \geq E(\boldsymbol{\alpha}) \quad \text{or} \quad E(\boldsymbol{\alpha}') \geq \Phi(\boldsymbol{\alpha}^*) \geq E(\boldsymbol{\alpha}^*) \geq E(\boldsymbol{\alpha}). \tag{31}$$

This cannot be true for all $\boldsymbol{\alpha}^*$'s which correspond to solutions of the boundary-value problem for different deformation paths because such paths include the minimum one, and then $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$, $\Phi(\boldsymbol{\alpha}) = E(\boldsymbol{\alpha})$, thus contradicting the first of (31) unless $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$. Since $\boldsymbol{\alpha}'$ may be chosen to be $\boldsymbol{\alpha}$ irrespective of $\boldsymbol{\alpha}^*$ the second possibility in (31) is also contradicted.

In Section 4 the results of the present section are transmitted from the local (micro-) level to the overall (macro-)level.

4. OVERALL VARIABLES AND INEQUALITIES

In dealing with the macroscopic (or overall) behaviour of a body consisting of the time-independent material in question, rather than with local phenomena, it is appropriate to work in terms of macroscopic (overall) variables.

We employ the (reference configuration) volume averages of $\boldsymbol{\alpha}$ and s, denoted by $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathrm{s}}$. Thus

$$\bar{\boldsymbol{\alpha}} = V^{-1}\int_V \boldsymbol{\alpha}\,\mathrm{d}V, \quad \bar{\mathrm{s}} = V^{-1}\int_V \mathrm{s}\,\mathrm{d}V \tag{32}$$

and, correspondingly, $\bar{\phi}$ is the volume average of ϕ the work done along any path to a given deformation from the reference configuration.

Under macroscopically uniform boundary conditions it can be shown that

$$\overline{\mathrm{s}\boldsymbol{\alpha}} = \bar{\mathrm{s}}\bar{\boldsymbol{\alpha}}, \quad \overline{\mathrm{s}\dot{\boldsymbol{\alpha}}} = \bar{\mathrm{s}}\dot{\bar{\boldsymbol{\alpha}}}, \quad \overline{\dot{\mathrm{s}}\dot{\boldsymbol{\alpha}}} = \dot{\bar{\mathrm{s}}}\dot{\bar{\boldsymbol{\alpha}}}, \tag{33}$$

([2, 19, 20] and, more generally, [13]) independently of the specific boundary conditions.

The volume average $\bar{\dot{\phi}}$ of the work increment $\dot{\phi} = \mathbf{s}\dot{\boldsymbol{\alpha}}$ can clearly be expressed as

$$\bar{\dot{\phi}} = \dot{\bar{\phi}} \quad (34)$$

with the help of $(33)_2$.

Thus, the average work done to a given deformation, $\overline{\phi(\boldsymbol{\alpha})}$, can be expressed as a function of $\bar{\boldsymbol{\alpha}}$ and is written $\bar{\phi}(\bar{\boldsymbol{\alpha}})$. Its value depends, of course, on the path taken to $\bar{\boldsymbol{\alpha}}$. The overall analogue of the work done on any path to $\boldsymbol{\alpha}$ locally is the work done on the corresponding macroscopic path to $\bar{\boldsymbol{\alpha}}$ relative to macroscopic variables:

$$\overline{\phi(\boldsymbol{\alpha})} \equiv \bar{\phi}(\bar{\boldsymbol{\alpha}}) = \int_0^{\bar{\boldsymbol{\alpha}}} \bar{\mathbf{s}}\,\mathrm{d}\bar{\boldsymbol{\alpha}}. \quad (35)$$

The volume average of eqn (14) now gives

$$\bar{\phi}(\bar{\boldsymbol{\alpha}}^*) + \overline{\phi_c(\mathbf{s}^*)} = \bar{\mathbf{s}}^*\bar{\boldsymbol{\alpha}}^*$$

with the help of $(33)_1$, and it follows that $\overline{\phi_c(\mathbf{s}^*)}$ can be written in terms of the average nominal stress $\bar{\mathbf{s}}^*$. Thus

$$\overline{\phi_c(\mathbf{s}^*)} \equiv \bar{\phi}_c(\bar{\mathbf{s}}^*) = \int_{\bar{\mathbf{s}}_0}^{\bar{\mathbf{s}}^*} \bar{\boldsymbol{\alpha}}\,\mathrm{d}\bar{\mathbf{s}}, \quad (36)$$

where $\bar{\mathbf{s}}_0$ is the average stress in the reference configuration. When the paths of deformation and stress are minimal and maximal respectively we have corresponding results for the comparison elastic material. For an elastic material the results have been given in [2, 20].

In particular, (35) specializes to

$$\bar{\mathbf{s}} = \frac{\partial \bar{W}}{\partial \bar{\boldsymbol{\alpha}}}, \quad (37)$$

$\bar{W}$ being a function only of $\bar{\boldsymbol{\alpha}}$.

For the comparison elastic material we have

$$\bar{W} + \bar{W}_c = \bar{\mathbf{s}}\bar{\boldsymbol{\alpha}} \quad (38)$$

and hence

$$\bar{\boldsymbol{\alpha}} = \frac{\partial \bar{W}_c}{\partial \bar{\mathbf{s}}} \quad (39)$$

uniquely in the overall counterpart of the domain $\mathscr{D}$.

Since the above formulae are independent of the boundary conditions [19, 20] the volume averages of inequalities (15) and (16) immediately yield

$$\bar{\phi}(\bar{\boldsymbol{\alpha}}) \geqslant \bar{W}(\bar{\boldsymbol{\alpha}}) \quad (40)$$

and

$$\bar{\phi}_c(\bar{\mathbf{s}}) \leqslant \bar{W}_c(\bar{\mathbf{s}}), \quad (41)$$

for all $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{s}}$. These may alternatively be obtained from (30) by putting $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$ and $\mathbf{s}^* = \mathbf{s}$.

The inequality (40) states that the overall work done in achieving average displacement gradient $\bar{\boldsymbol{\alpha}}$ in the material is bounded below, and (41) that the overall complementary work in achieving average nominal stress $\bar{\mathbf{s}}$ is bounded above. These are direct overall analogues of the local inequalities (15) and (16).

Now let the pair ($\mathbf{s}$, $\boldsymbol{\alpha}$) correspond specifically to the elasticity solution of the boundary-value problem for the comparison elastic material and ($\mathbf{s}^*$, $\boldsymbol{\alpha}^*$) a solution to the same boundary-value problem for the time-independent material, dependent of course on the deformation path.

For the first boundary-value problem, when $\mathbf{u}$ is specified on the whole of Σ, it is easily shown by transforming $(32)_1$ to an integral over Σ that

$$\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}^*. \tag{42}$$

But it does not in general follow that $\bar{\mathbf{s}} = \bar{\mathbf{s}}^*$ since $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{s}}$ are related by (37) while $\bar{\mathbf{s}}^*$ is the average nominal stress achieved on an arbitrary path to $\bar{\boldsymbol{\alpha}}^*$ in $\bar{\boldsymbol{\alpha}}$-space for the time-independent material.

In the second boundary-value problem $\mathbf{t}$ is prescribed on Σ and $(32)_2$ leads to

$$\bar{\mathbf{s}} = \bar{\mathbf{s}}^*, \tag{43}$$

but $\bar{\boldsymbol{\alpha}} \neq \bar{\boldsymbol{\alpha}}^*$ in general.

The chain of inequalities (30) establishes that

$$\bar{\phi}(\bar{\boldsymbol{\alpha}}^*) \geq \bar{W}(\bar{\boldsymbol{\alpha}}) \geq V^{-1}\int_\Sigma \mathbf{t}'\mathbf{u}_0\,\mathrm{d}\Sigma - V^{-1}\int_V W_c(\mathbf{s}')\,\mathrm{d}V \tag{44}$$

for the first boundary-value problem. In view of (42) the first inequality in (44) says no more than the overall characterization of minimum work (40). When the overall constitutive law of the comparison elastic material can be determined, which in principle is possible when the minimum work paths are known for a given time-independent material, we automatically have a lower bound on the overall work done to a specified overall displacement gradient for the considered material. Otherwise a weaker lower bound is obtainable in terms of the self-equilibrated nominal stress field $\mathbf{s}'$ in $\mathcal{D}_c$ and in respect of the comparison elastic material.

For the second boundary-value problem (30) gives

$$\bar{\phi}_c(\bar{\mathbf{s}}^*) \leq \bar{W}_c(\bar{\mathbf{s}}) \leq V^{-1}\int_V W_c(\mathbf{s}')\,\mathrm{d}V \tag{45}$$

which provides upper bounds for the overall complementary work, and, because of (43), the first of these bounds is consistent with (41).

It is particularly useful to consider the two special boundary-value problems above since $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{s}}$ are then known in terms of prescribed surface data for the first and second boundary-value problems respectively. For the mixed boundary-value problem, however, $\bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{s}}$ depend not only on the surface data but also on the particular form of constitutive law and the deformation path.

5. STRENGTHENING AND WEAKENING THEOREMS

The foregoing analysis has important implications in regard to the strengthening (or weakening) effects of a change of material properties. For example, one may wish to compare the overall response of a homogeneous material with that of a composite material whose matrix is of the same material but which contains voids, or inclusions of other material. In particular, we may wish to know whether the inhomogeneous material is stronger or weaker in some sense than the homogeneous material and whether or not it is stable (in some sense) under given loading conditions.

Definitions of strengthening and weakening of an elastic material have been discussed by Ogden[2] in respect of non-linear behaviour. These definitions are now used to provide definitions of strengthening and weakening for time-independent materials.

Let $\boldsymbol{\alpha}$ and $\mathbf{s}$ be related variables for the equilibrium of an elastic material with strain-energy function W per unit reference volume for the boundary-value problem under consideration. Without changing the initial geometry or boundary conditions suppose that the material properties are changed in such a way that the strain-energy function becomes W^+, and $\boldsymbol{\alpha}^+$ and $\mathbf{s}^+$ are the new displacement gradient and nominal stress variables.

According to [2] the material is said to be strengthened (locally) in the transition if

$$W(\boldsymbol{\alpha}^+) \leq W^+(\boldsymbol{\alpha}^+), \tag{46}$$

or, alternatively (but not in general equivalently),

$$W_c(\mathbf{s}) \geq W_c^+(\mathbf{s}). \tag{47}$$

Each of these local inequalities leads to

$$E(\boldsymbol{\alpha}) \leq E^+(\boldsymbol{\alpha}^+) \tag{48}$$

or, equivalently,

$$E_c(\mathbf{s}) \leq E_c^+(\mathbf{s}^+) \tag{49}$$

by means of (11), where E^+ and E_c^+ respectively are the energy and complementary-energy functionals for the W^+ material.

In terms of the overall constitutive laws the inequalities (48) and (49) respectively can be written

$$\bar{W}(\bar{\boldsymbol{\alpha}}) \leq \bar{W}^+(\bar{\boldsymbol{\alpha}}^+) + V^{-1} \int_\Sigma (\mathbf{u} - \mathbf{u}^+)\mathbf{t}\, d\Sigma$$

and

$$\bar{W}_c(\bar{\mathbf{s}}) \geq \bar{W}_c^+(\bar{\mathbf{s}}^+) + V^{-1} \int_\Sigma (\mathbf{t} - \mathbf{t}^+)\mathbf{u}\, d\Sigma.$$

And for the first and second boundary-value problems each of the above gives

$$\bar{W}(\bar{\boldsymbol{\alpha}}) \leq \bar{W}^+(\bar{\boldsymbol{\alpha}}^+), \quad \bar{W}_c(\bar{\mathbf{s}}) \geq \bar{W}_c^+(\bar{\mathbf{s}}^+) \tag{50}$$

respectively, where $\bar{\boldsymbol{\alpha}}^+ = \bar{\boldsymbol{\alpha}}$ and $\bar{\mathbf{s}}^+ = \bar{\mathbf{s}}$ for the first and second boundary-value problems respectively.

The first of these states that the total elastic stored energy of the body is increased when the material is strengthened under fixed boundary displacements. The second that the total complementary energy of the body is decreased when the body is strengthened under fixed boundary tractions. Some comments on the interpretation of this definition are made in Section 6. A corresponding definition of weakening is obtained by reversing the inequalities in (50) [2].

Suppose now that the properties of the time-independent material are changed so that the work done to achieve displacement gradient $\boldsymbol{\alpha}$, $\phi(\boldsymbol{\alpha})$, becomes $\phi^+(\boldsymbol{\alpha})$ along an arbitrary path in $\boldsymbol{\alpha}$-space. Let $W^+(\boldsymbol{\alpha})$ be the corresponding minimum of this, thus defining a new comparison elastic material.

The following definition is now introduced. *The time-independent material is said to be strengthened (weakened) if its comparison elastic material is strengthened (weakened).* That is, essentially, the material is strengthened (weakened) if the minimum work required to achieve a given deformation is increased (decreased) by the change of material. The strengthening-weakening effect (in terms of the present definition) in a time-independent material is therefore characterized by that in its comparison elastic material.

6. DISCUSSION

Current revived interest in the so-called "ideal strength" of perfect crystals has led to discussions of how the ideal strength should be defined (see, for example, the review by Macmillan [21]).

Approaching the subject through continuum mechanics Hill [22] has treated the concept of ideal strength as an instability phenomenon, but it is not clear what type of instability is appropriate. Hill has therefore considered the possibilities in terms of "dead" loading and "follower-force" loading relative to a general class of conjugate stress and strain measures. It is necessary to distinguish between different measures of stress and strain since the assumptions of linear elasticity are not valid.

A definition of ideal strength as instability under dead loading implies a dependence on geometry and type of loading, whereas relative to follower loading ideal strength is an *intrinsic material property* as pointed out by Hill[22].

At the macroscopic continuum level the definition of strengthening, as portrayed by inequalities (50) for elastic materials in Section 5 of this paper, is evidently an intrinsic property of the material if the macroscopic constitutive law characterizes the overall material independently of the boundary conditions. This is certainly the case when the boundary conditions are macroscopically homogeneous. It may be argued that a definition dependent on loading or geometry need not be ruled out if the boundary conditions are not of this type or the material is not a representative macroelement. Equations (50) are consistent with the intuitive notion that when a material is "strengthened' it requires a larger stress to break it.

Care should be taken when considering the *local* inequalities (46) and (47) because $\boldsymbol{\alpha}$ may be outside the stability domain for W^+ and could not then be used as the argument in place of $\boldsymbol{\alpha}^+$ in (46). A similar remark applies to $\mathbf{s}^+$ in respect of (47). Equations (46) and (47), on the other hand, remain valid when $\boldsymbol{\alpha}^+$ and s respectively are outside the stability domains for W and W_c^+.

The local definitions of strengthening (46) and (47) are intrinsic to the material since $W(\boldsymbol{\alpha})$ is independent of geometry and loading, and can equally well be expressed as a function of any other conjugate deformation or strain measure. The strengthening criteria considered in Section 5 throw no light on what definition of ideal strength might be adopted. However, they do ensure that, whatever conjugate measures are employed, the stress required to achieve a given deformation or strain (within the domain of stability relative to that measure) is "larger" for a stronger material than for a weaker one. A more detailed investigation of the relation between strengthening and stability is provided in [23].

For practical purposes a "real" material may be considered as a composite with a matrix of "ideal" material weakened by voids, cracks and inclusions. The strengthened, or "ideal" material is then regarded as being homogeneous in the matrix constituent of the original material. Comparison of experimental data for "real" and "ideal" specimens of a given material can then be made, having regard to (50).

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APPENDIX

We begin with the hypothesis that the work done, $\phi(\boldsymbol{\alpha})$, on an arbitrary path to any $\boldsymbol{\alpha}$ is bounded below, so that

$$\phi(\boldsymbol{\alpha}) \geqslant W(\boldsymbol{\alpha}). \tag{A1}$$

Let s be the terminal point in s-space of the trace of a minimum work path, and let $W_c(\mathbf{s})$ be the associated complementary work. Then

$$W(\boldsymbol{\alpha}) + W_c(\mathbf{s}) = \mathbf{s}\boldsymbol{\alpha}. \tag{A2}$$

Let $\phi_c(\mathbf{s})$ be the complementary work done on any path from the reference configuration to s, and let $\boldsymbol{\alpha}'$ be the terminus of the trace of this path in $\boldsymbol{\alpha}$-space. Suppose $\phi(\boldsymbol{\alpha}')$ denotes the work done on this path. Then

$$\phi(\boldsymbol{\alpha}') + \phi_c(\mathbf{s}) = \mathbf{s}\boldsymbol{\alpha}'. \tag{A3}$$

From (A2) and (A3) we obtain

$$W_c(\mathbf{s}) - \phi_c(\mathbf{s}) \geqslant W(\boldsymbol{\alpha}') - W(\boldsymbol{\alpha}) - \mathbf{s}(\boldsymbol{\alpha}' - \boldsymbol{\alpha})$$

on applying the hypothesis (A1) to $\boldsymbol{\alpha}'$. By convexity (19), it follows that this is positive. Hence, *the trace of a minimum-work path is a maximum complementary-work path.*

This result is valid in any domain where $W(\boldsymbol{\alpha})$ is strictly (globally) convex, even if the domain itself is not convex. On the other hand, starting from the hypothesis that the complementary work to any s is bounded above, the converse result that the trace of a maximum complementary work path is a minimum-work path can only be proved in a convex domain.